Perturbative Chern–Simons theory from the Penner model



2007 J. Phys. A: Math. Theor. 40 F443

(http://iopscience.iop.org/1751-8121/40/24/F02)

IOP FTC
IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) F443–F448 doi:10.1088/1751-8113/40/24/F02

**FAST TRACK COMMUNICATION**

# Perturbative Chern–Simons theory from the Penner model

**Noureddine Chair**

Physics Department, Al al-Bayt University, Mafraq, Jordan

E-mail: n.chair@rocketmail.com and nchair@aabu.edu.jo

Received 8 April 2007, in final form 11 May 2007
Published 30 May 2007
Online at stacks.iop.org/JPhysA/40/F443

**Abstract**
We show explicitly that the perturbative $SU(N)$ Chern–Simons theory arises naturally from two Penner models, with opposite coupling constants. As a result computations in the perturbative Chern–Simons theory are carried out using the Penner model, and it turns out to be simpler and transparent. It is also shown that the connected correlators of the puncture operator in the Penner model are related to the connected correlators of the operator that gives the Wilson loop operator in the conjugacy class.

PACS numbers: 11.15.Pg, 11.25.Pm
Mathematics Subject Classification: 14, D99

The free energy of the Penner model [1] is the generating function of the orbifold Euler characteristics of the moduli space of Riemann surfaces of genus $g$, with $s$ punctures. Computation of such a topological invariant was first computed by Harer and Zagier [2] by reducing a topological problem to a combinatorial problem and then solving it. The $SU(N)$ perturbative Chern–Simons free energy based on the $1/N$ expansion introduced by 't Hooft [3], and the Penner free energy have formally a similar topological expansion. This is the case, since both models use fatgraphs techniques that keep track of powers of $N$. The perturbative Chern–Simons free energy [4] may be written as

$$F = \sum_{g=0,h=1} C_{g,h} N^{2-2g} \lambda^{2g-2+h},$$

where $\lambda$ is the 't Hooft coupling constant and $h$ is the number of faces (boundaries) of the triangulated Riemann surfaces. In the Penner model $h$ is identical to the number of punctures. The coefficient $C_{g,h}$ was shown by Witten [5] to be identical to the partition function of the A-model topological open string theory at genus $g$ with $h$ boundaries on a six-dimensional target space $T^*S^3$. We will see that these coefficients $C_{g,h}$ are related to the orbifold Euler characteristics of the moduli space of Riemann surfaces of genus $g$, with $2h$ punctures.

  

The explicit expression for the Penner free energy $F = \log Z$ in terms of the genus and the punctures is [6]

$$F(t,N) = \sum_{g,s} N^{2-2g}(-1)^s t^{2g+s-2}\chi_{g,s}, \tag{1}$$

where the coefficients $\chi_{g,s}$ are the orbifold Euler characteristics of the moduli space of Riemann surfaces of genus $g$ with $s$ punctures; explicitly this topological invariant is given by

$$\chi_{g,s} = \frac{(-1)^s(2g-3+s)!(2g-1)}{(2g)!s!}B_{2g}$$

where $B_{2g}$ are the Bernoulli numbers. Note that the topological expansion of the free energy used by Distler and Vafa [7] is $F(t,N) = \sum_{g,s} N^{2-2g}t^{2-2g-s}\chi_{g,s}$. This expansion follows from equation (1) by simply letting $t \to -\frac{1}{t}$.

Let us now consider a sum of two Penner models, one with a coupling constant $t$ and the other with a coupling constant $-t$, such that the topological expansion of the free energy in both cases is given by equation (1). If the coupling constant in the Penner model is set to be equal to $\lambda/2\pi n$, $\lambda$ is the Chern–Simons Coupling constant and $n$ is a positive integer. Let $F(\lambda, N)$ be the total free energy for the two Penner models, then by using equation (1) and summing over $n$ one has

$$\begin{aligned} F(\lambda,,N) &= \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty} N^{2-2g}(\lambda/2\pi n)^{2g+s-2}\chi_{g,s}((-1)^s+1) \\ &= 2\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} N^{2-2g}(\lambda/2\pi n)^{2g+2p-2}\chi_{g,2p}. \end{aligned} \tag{2}$$

Explicitly, if we write $F(\lambda, N) = \sum_g N^{2-2g}F_g(\lambda)$, then the genus-$g$ contribution $F_g(\lambda)$ to the free energy is nothing but the perturbative $SU(N)$ Chern–Simons free energy $F_g^p(\lambda)$ on $S^3$ [4],

$$\begin{aligned} F_0(\lambda) &= -2\sum_{n=1}^{\infty}\sum_{p=2}^{\infty}\frac{1}{2(p-1)2p(2p-1)}\left(\frac{\lambda}{2\pi n}\right)^{2p-2} \\ F_1(\lambda) &= \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{B_2}{2p}\left(\frac{\lambda}{2\pi n}\right)^{2p} \\ F_g(\lambda) &= 2\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{(2g-3+2p)!(2g-1)}{(2g)!(2p)!}B_{2g}\left(\frac{\lambda}{2\pi n}\right)^{2g+2p-2}. \end{aligned} \tag{3}$$

Note that the Bernoulli numbers $B_g$ in the above equation are alternating unlike those in [4], are taken to be all positive. As one can see from the above equation, the computations are simple and follow immediately from the Penner model. The genus expansion of the free energy in the perturbative Chern–Simons theory [4] is obtained from the following perturbative term in $\lambda$,

$$F^p(\lambda) = \sum_{j=1}^{N-1}(N-j)\sum_{n=1}\ln\left(1-\frac{j^2\lambda^2}{4\pi^2 n^2 N^2}\right).$$

Having identified the perturbative Chern–Simons free energy $F_g^p(\lambda)$ on $S^3$ with the extended Penner model described above, we may use the latter to do our computations in the perturbative Chern–Simons. Here computations are done for both the double-scaling limit

[8], and summation over the boundaries $p$ as well as over the integer $n$ of the perturbative Chern–Simons [9]. Let us first find the continuum limit in this theory; to do so, one needs to sum over all faces (boundaries) $p$ in the free energy $F_g^p(\lambda)$. Using the Penner model, this summation is equivalent to sum over all punctures. For $g = 0$, Penner model the sum over all punctures was computed explicitly [6], and we obtained the following identity,

$$F_0(t) = -\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} t^k = \frac{(1-t)^2}{2t^2} \ln(1-t) - \frac{3}{4} + \frac{1}{2t}. \tag{4}$$

Therefore, from the relation between the perturbative Chern–Simons and the extended Penner model summarized by equation (2), the sum over boundaries for $g = 0$ is

$$F_0 = \sum_{n=1}^{\infty} \left[ \frac{(1-\lambda/2\pi n)^2}{2(\lambda/2\pi n)^2} \ln(1-\lambda/2\pi n) + \frac{(1+\lambda/2\pi n)^2}{2(\lambda/2\pi n)^2} \ln(1+\lambda/2\pi n) - \frac{3}{2} \right]. \tag{5}$$

The free energy is even in both $n$, and $\lambda$ as it should be, see equation (3). If we define a new coupling constant $\nu_n$ by $\nu_n = \frac{2\pi N}{\lambda}\left[\frac{\lambda}{2\pi} - n\right]$, as in [8], then multiplying the above sum by $N^2$ gives

$$\begin{aligned} N^2 F_0(\lambda) = {} & \sum_{n=1}^{\infty} \left[ \frac{\mathrm{i}\pi}{2} \nu_n^2 + \frac{1}{2} \nu_n^2 \ln(\nu_n/N) - \frac{3}{4} \nu_n^2 + (n \to -n) \right] \\ & - N^2 \sum_{n=1}^{\infty} \left[ \ln\left(\frac{2\pi n}{\lambda}\right) + \frac{4\pi^2 n^2}{\lambda^2} \left( \ln(\lambda/2\pi n) \left(\frac{2\pi n}{\lambda}\right) - \frac{3}{2} \right) \right]. \end{aligned} \tag{6}$$

Note the presence of the term, $\frac{\mathrm{i}\pi}{2}\nu_n^2$, is responsible for the evenness of the free energy with respect to $n$ and $\lambda$. For $g = 1$, the summation over boundaries is simpler and one may use either $F_1(\lambda)$ given by equation (3), or the Penner free energy [6] given by $F_1(\lambda) = -\frac{B_2}{2} \ln(1-t)$. Therefore, in this case one has

$$F_1 = -\frac{1}{2} B_2 \sum_{n=1}^{\infty} \left[ \mathrm{i}\pi + \ln\left(\frac{\nu_n}{N}\right) + (n \to -n) \right] + B_2 \sum_{n=1}^{\infty} \ln\left(\frac{2\pi n}{\lambda}\right). \tag{7}$$

Before summing over boundaries for the higher genus $g \geqslant 2$, one gives first summation over punctures in the Penner model. Using the identity $-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2g-3} t^{-(s+1)} = \frac{(2g-3+s)!}{s!} t^{2-2g-s}$, the sum over punctures in the free energy $F_g(t) = \sum_s t^{2-2g-s} \chi_{g,s}$ for $g \geqslant 2$ is

$$F_g(t) = \left[ \frac{1}{(1+t)^{2g-2}} - \frac{1}{t^{2g-2}} \right] \chi_{g,0}, \tag{8}$$

where $\chi_{g,0} = \frac{B_{2g}}{2g(2g-2)}$ is the orbifold Euler characteristic of the moduli space without punctures. Incidentally, the topological expansion for $F_g(t)$ studied in [7] has two critical points, namely $t = -1$ and $t = 0$. As we made it clear in this paper, the perturbative Chern–Simons is connected to the original topological expansion of the Penner model studied in [6], i.e., equivalent to letting $t \to -\frac{1}{t}$ in equation (8). Therefore the sum over the punctures in the original Penner model reads

$$F_g(t) = [(1-1/t)^{2-2g} - t^{2g-2}] \chi_{g,0}. \tag{9}$$

Now, as we did for $g = 0$ and $g = 1$, the sum over the boundaries in the perturbative Chern–Simons for $F_g(\lambda)$ is the sum over $n$ of $F_g\left(\frac{\lambda}{2\pi n}\right) + F_g\left(-\frac{\lambda}{2\pi n}\right)$, that is,

$$F_g(\lambda) = \chi_{g,0} \sum_{n=1}^{\infty} \left[ \nu_n^{2-2g} + (n \to -n) \right] - 2\chi_{g,0} \left( \frac{\lambda}{2\pi N} \right)^{2g-2} \zeta(2g-2), \tag{10}$$

where $\zeta(2g-2)=\sum_{n=1}^{\infty}\frac{1}{n^{2g-2}}$. The Chern–Simons theory coupling constant $\lambda$ is related to the level of the Kac–Moody algebra $k$ by $\lambda=\frac{2\pi N}{k+N}$; this shows that $\lambda$ has a fundamental domain between 0 and $2\pi$. Therefore, the natural critical double scaling limit would be

$$\lambda\to 2\pi \qquad \nu_1=\text{finite}.$$

Note that at this critical point $\lambda/2\pi\to 1$ that is, $t\to 1$ which is nothing but the critical point in the Penner model [6].

We turn now to the sum over the boundaries $p$, and over $n$ of the free energy $F_g(\lambda)$; this sum is known to be connected to the Hodge integrals and Gromov–Witten theory [10]. For $g=0$, the computations are easy and straightforward. From equation (5) one has

$$\left(\frac{N}{\lambda}\right)^2F_0(\lambda)=\left(\frac{N}{\lambda}\right)^2\sum_{n=1}^{\infty}\left[4\pi^2n^2\frac{(1-\lambda/2\pi n)^2}{2}\ln(1-\lambda/2\pi n)\right.$$
$$\left.+4\pi^2n^2\frac{(1+\lambda/2\pi n)^2}{2}\ln(1+\lambda/2\pi n)-\frac{3\lambda^2}{2}\right]. \tag{11}$$

Differentiating $F_0(\lambda)$, with respect to $\lambda$, twice gives $\mathrm{d}^2/\mathrm{d}\lambda^2F_0(\lambda)=\sum'_{n\in\mathbf{Z}}[\ln(1-\lambda/2\pi n)]$, where $n=0$ is not included in the sum. From the product formula $\frac{\sin(\pi x)}{\pi x}=\prod_{n=1}\left(1-\frac{x^2}{n^2}\right)$, one has the identity

$$\sum_{n\in\mathbf{Z},n\neq 0}[\ln(1-\lambda/2\pi n)]=\mathrm{i}\lambda/2+\ln(1-\mathrm{e}^{-\mathrm{i}\lambda})-\ln\lambda-\mathrm{i}\frac{\pi}{2}. \tag{12}$$

Therefore, we see that our computations using the Penner model are simpler and transparent. The summed free energy $F_0(\lambda)$ is obtained simply by integrating twice the expression for $\mathrm{d}^2/\mathrm{d}\xi^2F_0(\xi)$ with respect to $\xi$ from 0 to $\lambda$. Note that here we follow closely the same lines in deriving the product formula for $\sin x$. Taking $\ln\lambda=2\mathrm{i}\pi m$, $\theta=0$, the cut line being the real axis then

$$F_0(\lambda)=\zeta(3)+\mathrm{i}\zeta(2)-\mathrm{i}(m+1/4)\pi\lambda^2+\mathrm{i}\lambda^3/12+\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}n\lambda}}{n^3}; \tag{13}$$

this is identical to the result obtained in [9]. The coefficient of the last term in the above equation is the $g=0$ Gromov–Witten invariant of a Calabi–Yau 3-fold, $C(0,n)=\frac{1}{n^3}$ [10]. For $g=1$ one can see that the free energy up to constant terms is given by

$$F_1(\lambda)=\sum_{n\in\mathbf{Z},n\neq 0}[\ln(1-\lambda/2\pi n)]=-B_2/2[\mathrm{i}\lambda/2+\ln(1-\mathrm{e}^{-\mathrm{i}\lambda})]. \tag{14}$$

The Gromov–Witten invariant in this case follows from the second term and is given by $C(1,n)=\frac{1}{12n}$.

Now, we come to the sum over boundaries $p$ for $g\geqslant 2$, as we explained before this is equivalent to summing over punctures in the proposed penner model. By rewriting equation (10), explicitly in terms of $\lambda$, we have

$$F_g(\lambda)=\left(\frac{N}{\lambda}\right)^{2-2g}\chi_{g,0}\sum_{n\in\mathbf{Z},n\neq 0}[(\lambda-2\pi n)^{2-2g}]-2\left(\frac{N}{\lambda}\right)^{2-2g}\chi_{g,0}\left(\frac{1}{2\pi}\right)^{2g-2}\zeta(2g-2). \tag{15}$$

The sum over $n$ may be carried out by simply using the product formula for $\sin\pi x$, from which we obtain $\sum_{n\in\mathbf{Z}}\ln(\lambda-2\pi n)\approx\ln(1-\mathrm{e}^{-\mathrm{i}\lambda})$. Note that the terms that are not written would disappear upon differentiating $(2g-2)$ times the right-hand side of the approximation, and so we have

$$F_g(\lambda) = \left(\frac{N}{\lambda}\right)^{2-2g} (-1)^{g-1}\chi_{g,0}\frac{1}{(2g-3)!}\sum_{n\geqslant 1} n^{2g-3}\,\mathrm{e}^{-\mathrm{i}n\lambda} - 2\left(\frac{N}{\lambda}\right)^{2-2g}\chi_{g,0}\left(\frac{1}{2\pi}\right)^{2g-2}\zeta(2g-2). \tag{16}$$

The coefficient of the first term may be written as $|\chi_{g,0}|\frac{n^{2g-3}}{(2g-3)!}$, which is nothing but the Gromov–Witten invariant $C(g,n)$ [10]. The coefficient of the second term $2\chi_{g,0}\left(\frac{1}{2\pi}\right)^{2g-2}\zeta(2g-2)$ is the degree zero Gromov–Witten invariant [9]. This is identical to the Hodge integral, $\int_{\mathcal{M}_g} c_{g-1}^3$ [4, 10], where $c_{g-1}$ is the $(g-1)$ Chern class of the Hodge bundle.

We now push further the connection between the Penner model and the perturbative Chern–Simons theory. We carry out this connection by considering correlators in the Penner model that exhibit logarithmic singularities, and find out the corresponding correlators in the CS theory. In the former the correlators that exhibit logarithmic scaling violation are the puncture operators $\partial/\partial\mu$ [6, 7, 11], where $\mu = N(1-t)$. The free energy $F_g(\mu)$ is known to be related to the orbifold Euler characteristics $\chi_{g,0}$. When differentiated $s$-times, we obtain a generating function for the orbifold Euler characteristics with $s$ punctures. This procedure is equivalent to putting $s$ punctures on a Riemann surface. We have shown in [11] that the puncture operator $\partial/\partial\mu$ is identified with the operator $\mathrm{Tr}\ln(1-\mathrm{i}\sqrt{t}M)$, where $M$ is an $N\times N$ Hermitian matrix. The equivalence of the two operators was checked for the one-point and the two-point connected correlators. Explicitly the $k$th power for the operator $\mathrm{Tr}\ln(1-M)$ reads

$$\begin{aligned}\frac{1}{k!}(\mathrm{Tr}\ln(1-M))^k &= (-1)^k\left(\sum_{j\geqslant 1}\mathrm{Tr}\frac{M^j}{j}\right)^k \\ &= (-1)^k\frac{1}{k!}\sum_{k_1\geqslant 0,k_2\geqslant 0,\ldots}\frac{k!}{k_1!k_2!\cdots}\left(\frac{\mathrm{Tr}\,M}{1}\right)^{k_1}\left(\frac{\mathrm{Tr}\,M^2}{2}\right)^{k_2}\cdots \\ &= (-1)^k\sum_{k_1\geqslant 0,k_2\geqslant 0,\ldots}\frac{1}{\prod_{j\geqslant 1}k_j!\,j^{k_j}}\prod_{j=1}^{\infty}(\mathrm{Tr}\,M^j)^{k_j},\end{aligned} \tag{17}$$

where the sum is taken over all $k$'s, zero or positive integers such that $\sum k_j = |\vec{k}| = k$. Therefore, the connected $k$th correlators of the puncture operator may be written as

$$\frac{1}{k!}\langle(\mathrm{Tr}\ln(1-M))^k\rangle^c = (-1)^k\sum_{k_1\geqslant 0,k_2\geqslant 0,\ldots}\frac{1}{z_{\vec{k}}}\langle\Upsilon_{\vec{k}}(M)\rangle^c, \tag{18}$$

where $z_{\vec{k}} = \prod_{j\geqslant 1}k_j!\,j^{k_j}$, $\Upsilon_{\vec{k}}(M) = \prod_{j=1}^{\infty}(\mathrm{Tr}\,M^j)^{k_j}$; this formally looks like the operator that gives the Wilson loop operator in the conjugacy class basis [12]. Therefore the correlators of the puncture operator in the Penner model are very close to the logarithm of the expectation of the Ooguri–Vafa operator [12, 13] $\ln\langle Z(U,V)\rangle = \sum_{\vec{k}}\frac{1}{z_{\vec{k}}}\langle\Upsilon_{\vec{k}}(U)\rangle^c\Upsilon_{\vec{k}}(V)$. When the $M\times M$ matrix $V$ (source term) is the unit matrix then formally correlators of the puncture operator and the above generating function are identical up to a constant factor. Therefore, it is possible to compute $\langle\Upsilon_{\vec{k}}(U)\rangle^c$ from the connected correlators of the Penner model [6]. These correlators are written in terms of the orbifold Euler characteristics with punctures.

Finally we point out that the connection between the perturbative Chern–Simons theory and the Penner model may also be seen from the explicit expression for the free energy of the Penner model [6],

$$F(t,N)=\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)}\left(\frac{t}{N}\right)^{2m-1}+\sum_{p=1}^{N-1}(N-p)\ln\left(1-\frac{pt}{N}\right).$$

To obtain the perturbative Chern–Simons free energy, we follow the same procedure used in the paper. Let $t\to -t$, then one has

$$F(t,N)+F(-t,N)=\sum_{p=1}^{N-1}(N-p)\ln\left(1-\frac{p^2t^2}{N^2}\right).$$

Next, set $t=\frac{\lambda}{2\pi n}$, then summing over $n\geqslant 1$ gives the perturbative Chern–Simons free energy [4]. It remains to see the physical justification and interpretation of the connection between the perturbative Chern–Simons theory and the extended Penner model, proposed in this paper.

## Acknowledgments

I would like to thank M Marino for reading and making comments on this work. I would also like to thank G Bonelli, L Bonora, K S Narain and E H Saidi for discussions during the early stages of this work, H M Al-Nasser and M I Serhan for their useful remarks, Al al-Bayt university, the Abdus Salam center ICTP, for the support.

## References

[1] Penner R C 1998 *J. Diff. Geom.* **27** 35
[2] Harer J and Zagier D 1986 *Invent. Math.* **85** 457
[3] ’t Hooft G 1974 *Nucl. Phys.* B **72** 461
[4] Gopakumar R and Vafa C 1998 M-theory and topological strings, I *Preprint* hep-th/9812127
[5] Witten E 1995 *Prog. Math.* **133** 637
[6] Chair N 1991 *Rev. Math. Phys.* **3** 285
[7] Distler J and Vafa C 1991 *Mod. Phys. Lett.* A **6** 259
[8] Das S R and Gomez C 2004 *J. High Energy Phys.* JHEP10(2004)026
[9] Gopakumar R and Vafa C 1999 *Adv. Theor. Math. Phys.* **3** 1415
[10] Faber C and Pandharipande R 2000 *Invent. Math.* **139** 173
[11] Chair N and Panda S 1991 *Phys. Lett.* **272B** 230
[12] Marino M 2005 *Rev. Mod. Phys.* **77** 675
[13] Ooguri H and Vafa C 2000 *Nucl. Phys.* B **577** 419